Math 2058, HW 2. Due: 8 Oct 2024, before 11:59 pm

(1) If S is a non-empty subset of $\mathbb{R}$ which is bounded from above but not below. Suppose the following holds: $[x, y] \subset S$, for all $x, y \in S$. Show that S is either $(-\infty, \alpha]$ or $(-\infty, \alpha)$ for some $\alpha \in \mathbb{R}$.

(2) Using ε-N terminology, show the followings:
  (a) $\lim_{n\to+\infty} \frac{n}{n^2-2} = 0$.
  (b) $\lim_{n\to+\infty}(2n)^{1/n} = 0$.
  (c) $\lim_{n\to+\infty} 2^n/n! = 0$.

(3) Suppose (x_n) is a sequence of positive real number such that $\lim_{n\to+\infty} x_{n+1}/x_n = L \in \mathbb{R}$.
  (a) Show that (x_n) is convergent if $L \in [0, 1)$.
  (b) Can we conclude the convergence if $L = 1$? Justify your answer.

(4) If $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}$$

for all $n \in \mathbb{N}$. Show that (x_n) is convergent and $x_n < 2$ for all $n \in \mathbb{N}$.

(5) If $x_n = \sum_{k=1}^{n} a_k$ for some sequence (a_k). Suppose (x_n) is convergent, and (b_k) is another sequence of positive real number which is monotonic increasing and bounded, show that $y_n = \sum_{k=1}^{n} a_k b_k$ is convergent.